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Dynamical boundary conditions for integrable lattices

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Abstract. We introduce a natural extension of the usual covariance property for the reflection equation. To solve one reflection equation we propose to use a family of reflection equations with various Drinfeld twists of the initial R-matrix. This allows us to produce non-trivial representations of the reflection equation algebra in a systematic way. As an example, we consider a twist of the Lie algebra $sl(2)$ related to some integrable tops and to the Toda lattices associated with the $\mathcal{D}_n$ root system.

1. Introduction

One of the cornerstones of the quantum inverse scattering method is quadratic R-matrix algebra with spectral parameter dependence. The rich structure of the Yang–Baxter algebra [1, 2] and of reflection equation algebra [4, 7] permits us to include into this formalism a variety of known integrable models and to find new ones.

However, to find a representation of these algebras in a given quantum space is a formidable task unless we have some additional information. In particular, if we know a representation of the Yang–Baxter algebra, we can construct representations of the reflection equation algebra by using the covariance property of this algebra [4, 7]. In terms of integrable models the representations of the Yang–Baxter algebra provide us with integrable lattices while the $\mathbb{C}$ number representations of the reflection equation algebra describe the possible boundary conditions for such lattices [3–6].

In this paper we consider generalizations of the covariance property for the reflection equation. In this way the reflection equation is reduced to a system of equations, which allows us to produce non-trivial representations of the reflection equation algebra in a systematic way. For a given integrable lattice the proposed system of equations may be solved by using either known Drinfeld twists of the R-matrix or some ansatz for a boundary matrix. We would like to stress that the resulting boundary matrices act in the same quantum space with the initial representation of the Yang–Baxter algebra. It allows us to comment upon dynamical boundary conditions for integrable lattices. From a more mathematical point of view an interesting relation between the Drinfeld twist and the associated twist of the underlying Lie algebra is made. We shall start with several abstract propositions and then discuss its applications to different integrable lattices. As an example, we present dynamical boundary conditions associated to the twist of the algebra $sl(2)$ for the Toda lattice and an integrable top.

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2. On a covariance property of the reflection equation

Let the operator-valued function $R(u) : \mathbb{C} \to \mathrm{Aut}(V \otimes V)$ be a solution to the quantum Yang–Baxter equation [1] in a finite-dimensional linear space V. Let us connect with a given matrix $R(u)$ two associative algebras $\mathcal{T}_R$ and $\mathcal{U}_R$ generated by non-commutative entries of the square matrices $T(u)$ and $K(u)$ satisfying the fundamental commutator relation [1, 2]

$$R_{12}(u-v)\overset{1}{T}(u)\overset{2}{T}(v) = \overset{2}{T}(v)\overset{1}{T}(u)R_{12}(u-v) \tag{2.1}$$

or the reflection equation [3, 4]

$$R_{12}(u-v)\overset{1}{K}(u)R_{21}(u+v)\overset{2}{K}(v) = \overset{2}{K}(v)R_{12}(u+v)\overset{1}{K}(u)R_{21}(u-v). \tag{2.2}$$

Here $\overset{1}{X} \equiv X \otimes \mathrm{id}_{V_2}$, $\overset{2}{X} \equiv \mathrm{id}_{V_1} \otimes X$ for any matrix $X \in \mathrm{End}(V)$. As usual, $R_{ij}(u) \in \mathrm{End}(V_i \otimes V_j)$, $V_j \equiv V$ and $R_{ji}(u) = PR_{ij}(u)P$, with P as the permutation operator in the tensor product of the two spaces $V_i \otimes V_j$ [1, 2].

In this paper we discuss solutions to the reflection equation (2.2), i.e. consider various representations of the algebra $\mathcal{U}_R$ [4]. It is known that the covariance property of the reflection equation (2.2) [4, 7] may be used to construct new solutions starting from known ones. The following was pointed out in [4].

Proposition 1. Let the matrices $T(u)$ and $K(u)$ satisfy the relations (2.1) and (2.2) with the same R-matrix $R(u)$, then the Sklyanin monodromy matrix

$$K'(u) = T(u)K(u)T^{-1}(-u) \tag{2.3}$$

solves the reflection equation (2.2) if

$$\overset{1}{T}(u)\overset{2}{K}(v) = \overset{2}{K}(v)\overset{1}{T}(u). \tag{2.4}$$

The proof follows easily by substitution of $K'(u)$ into the reflection equation (2.2) and by using a few different forms of the fundamental relation (2.1), for example

$$\overset{2}{T}{}^{-1}(-v)R_{12}(u+v)\overset{1}{T}(u) = \overset{1}{T}(u)R_{12}(u+v)\overset{2}{T}{}^{-1}(-v). \tag{2.5}$$

The main condition (2.4) holds if $T(u)$ and $K(u)$ are representations of the algebras $\mathcal{T}_R$ and $\mathcal{U}_R$ in the different spaces $\mathcal{H}_1 \otimes V$ and $\mathcal{H}_2 \otimes V$ respectively, hence the entries $T(u)$ and $K(u)$ mutually commute. Here $\mathcal{H}_1$ and $\mathcal{H}_2$ are the distinct quantum spaces and V is a common auxiliary space [1, 2].

In this paper the Sklyanin monodromy matrix K' (2.3) is constructed from the known representations $T(u)$ and $K(u)$ of the algebras $\mathcal{T}_R$ and $\mathcal{U}_R$ defined on the common quantum space $\mathcal{H}$, i.e. using matrices with the non-ultralocal commutation relations [8]. In this case the initial matrix $K(u)$ will be termed the *dynamical* boundary matrix.

Let us include the construction of the Sklyanin representation K' (2.3) and ignore the covariance property of the initial reflection equation (2.2).

Proposition 2. Let $T(u)$ satisfy the fundamental commutator relation (2.1). If it is intertwined with the dynamical boundary matrix $K(u)$, the initial matrix $R(u)$ and by some new matrix $S(u, v)$

$$\overset{1}{K}(u)\overset{2}{T}(v)R_{12}(u+v)\overset{1}{T}{}^{-1}(-u)\overset{2}{K}(v) = \overset{2}{T}(v)\overset{1}{K}(u)S_{21}(u,v)\overset{2}{K}(v)\overset{1}{T}{}^{-1}(-u) \tag{2.6}$$

then the Sklyanin monodromy matrix

$$K'(u) = T(u)K(u)T^{-1}(-u)$$

solves the initial reflection equation (2.2) provided that the dynamical boundary matrix $K(u)$ satisfies the generalized reflection equation

$$R_{12}(u-v)\overset{1}{K}(u)S_{21}(u,v)\overset{2}{K}(v) = \overset{2}{K}(v)S_{12}(u,v)\overset{1}{K}(u)R_{21}(u-v). \tag{2.7}$$

Since $S(u, v)$ is an arbitrary matrix the proof consists of the direct verification of the reflection equation (2.2) by using relations (2.1), (2.6) and (2.7).

For a given matrix $T(u)$ we try to find the dynamical boundary matrix $K(u)$ together with the matrix $S(u, v)$ from equation (2.6), then we have to check the generalized reflection equation (2.7) intertwining both these matrices with the R-matrix.

Particular solutions to equation (2.6) may be obtained from the following system of equations

$$\begin{aligned} &\overset{1}{K}(u)\overset{2}{T}(v) = \overset{2}{T}(v)\overset{1}{K}(u)F(u,v) \\ &\overset{1}{T}{}^{-1}(-u)\overset{2}{K}(v) = G(u,v)\overset{2}{K}(v)\overset{1}{T}{}^{-1}(-u) \end{aligned} \tag{2.8}$$

with the two unknown matrices $F(u, v)$ and $G(u, v)$. In this case the dynamical boundary matrix $K(u)$ has to be a solution to the generalized reflection equation (2.7) with the matrix

$$S(u,v) = F(u,v)R(u+v)G(u,v). \tag{2.9}$$

Equations (2.8) have the same form as exchange algebras [9]. Obviously, other forms of these algebra are

$$\begin{aligned} &F(u,v) = \overset{1}{K}{}^{-1}(u)\overset{2}{T}{}^{-1}(v)\overset{1}{K}(u)\overset{2}{T}(v) \\ &G(u,v) = \overset{1}{T}{}^{-1}(-u)\overset{2}{K}(v)\overset{1}{T}(-u)\overset{2}{K}{}^{-1}(v). \end{aligned}$$

Let the dynamical boundary matrix $K(u)$ have the following property

$$K(u)K(-u) = \phi(u)I \tag{2.10}$$

where $\phi(u)$ means some scalar function. In this case the matrix $S(u, v)$ (2.9) is the Drinfeld twist [10] of the matrix $R(u + v)$

$$S(u,v) = F(u,v)R(u+v)F_{21}^{-1}(-v,-u) \qquad F_{21}(u,v) = PF(u,v)P \tag{2.11}$$

if the universal twist element $F(u, v)$ has appropriate properties [10]. Note, a twist transformation (2.11) of the R-matrix related to the braid group $\check{R} = PR$

$$\check{S}(u,v) = PS(u,v) = F_{21}(u,v)\check{R}F_{21}^{-1}(-v,-u)$$

looks just like Sklyanin formulae (2.3), in contrast to the usual similarity transformation in quantum group theory.

We next recall that, for integrable lattice models, matrix $T(u)$ is constructed as an ordered product

$$T(u) = L_n(u)L_{n-1}(u)\dots L_1(u) \tag{2.12}$$

of n independent L-operators having some simple dependence on the spectral parameter u. Matrices $L_k(u)$ in the chain (2.12) are operators which act on the different local spaces $\mathcal{H}_k \otimes V$, we now use the notation $\mathcal{H}_k$ for a local quantum space assigned to the site k in the lattice and $V \simeq \mathbb{C}^n$ is a common auxiliary space [1, 2].

Let $K(u)$ be the representation of the reflection equation algebra $\mathcal{U}_R$ in the space $\mathcal{H}_-\otimes V$. Then, the Sklyanin monodromy matrix (2.3) [4–6]

$$K_-(u) = L_n(u)L_{n-1}(u)\dots L_1(u)K(u)L_1^{-1}(-u)L_2^{-1}(-u)\dots L_n^{-1}(-u) \tag{2.13}$$

describes a lattice model with boundary conditions corresponding to the matrix $K(u)$. As usual, the transfer matrix is given by

$$\tau(u) = \mathrm{tr}K_+(u)T(u)K(u)T^{-1}(-u) = \mathrm{tr}K_+(u)K_-(u). \tag{2.14}$$

Here an extra boundary K-matrix K_+ is some solution of a 'conjugated' reflection equation [4, 7] on the quantum space $\mathcal{H}_+$ defined in such a way to guarantee the commutativity $[\tau(u), \tau(v)] = 0$. This transfer matrix gives rise to the Hamiltonian and other integrals of motion for a quantum system with the space of states $\mathcal{H} = \mathcal{H}_+\otimes\mathcal{H}_n\otimes\mathcal{H}_{n-1}\dots\mathcal{H}_1\otimes\mathcal{H}_-$.

Looking for dynamical boundary matrices $K(u)$ we can start with a single operator $L(u)$ in the chain (2.13). In addition, we can begin either with the generalized reflection equation (2.7) by using known twists $S(u, v)$ [10, 12, 16], or with the exchange algebras (2.8) by assuming some ansatz for the boundary matrix $K(u)$.

3. The Toda lattices

As an example, let us consider the following L-operator

$$L(u) = \begin{pmatrix} u-p & -\exp(q) \\ \exp(-q) & 0 \end{pmatrix} \qquad [p, q] = -\mathrm{i}\eta \tag{3.1}$$

where (p, q) is a pair of canonical conjugated variables. This L-operator is intertwined (2.1) by the rational Yang R-matrix

$$R(u) = uI - \mathrm{i}\eta P \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

where P is the permutation operator in $\mathbb{C}^2\otimes\mathbb{C}^2$ and

$$R(u) = \begin{pmatrix} u-\mathrm{i}\eta & 0 & 0 & 0 \\ 0 & u & -\mathrm{i}\eta & 0 \\ 0 & -\mathrm{i}\eta & u & 0 \\ 0 & 0 & 0 & u-\mathrm{i}\eta \end{pmatrix} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g & h & 0 \\ 0 & h & g & 0 \\ 0 & 0 & 0 & f \end{pmatrix}(u). \tag{3.3}$$

In this case the monodromy matrix $T(u)$ (2.12) describes the Toda lattices associated with the root system $\mathcal{A}_n$ [18]. The corresponding Hamiltonian reads as

$$H_{\mathrm{A}} = \sum_{j=1}^{n}\frac{1}{2}p_j^2 + \sum_{j=1}^{n-1}\exp(q_{j+1}-q_j). \tag{3.4}$$

The set of operators $\{L(q_j, p_j, u)\}_{j=1}^n$ (3.1) defines the monodromy matrix $T(u)$ (2.12), which is a spin-1/2 representation of the Yangian $Y(sl(2))$ in $V = \mathbb{C}^2$. So, the inversion $T^{\mathrm{i}}(u) = T^{-1}(-u)$ is equal to [4]

$$T^{\mathrm{i}}(u) = \frac{\sigma_2 T^{\mathrm{t}}(-u-\mathrm{i}\eta)\sigma_2}{\Delta\{T(-u-(\mathrm{i}\eta/2))\}}. \tag{3.5}$$

where t means matrix transposition, σ_2 is the Pauli matrix and $\Delta\{T(u)\}$ is a quantum determinant of $T(u)$. According to the general recipe [4], it allows us to work with the algebra $\tilde{\mathcal{U}}_R$ instead of $\mathcal{U}_R$. The new algebra $\tilde{\mathcal{U}}_R$ has the following Sklyanin representation

$$K_- = T(u)K\left(u - \frac{\mathrm{i}\eta}{2}\right)\sigma_2 T^{\mathrm{t}}(-u)\sigma_2. \tag{3.6}$$

Let us begin with the known scalar solution of the reflection equation (2.2)

$$K_{\mathrm{c}}(u) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(u) \equiv \begin{pmatrix} \alpha u + \delta & \beta u \\ \gamma u & -\alpha u + \delta \end{pmatrix} \tag{3.7}$$

where α, β, γ and δ are complex numbers. By using the monodromy matrix $T(u)$ (2.12), the usual covariance property and the solution $K_{\mathrm{c}}(u)$ (3.7) one can get the monodromy matrix K_- (3.6), which describes the Toda lattices associated with the root system $\mathcal{A}_n$ by $\alpha = \delta = \gamma = 0$ (3.4) and with the root system $\mathcal{BC}_n$ by $\beta = 1$ [4, 5]. In the second case the corresponding Hamiltonian is given by

$$H_{BC} = H_{\mathrm{A}} - \frac{\gamma}{2}\exp(2q_1) + (\delta - \alpha p_1)\exp(q_1). \tag{3.8}$$

The transfer matrix (2.14) with the 'conjugated' to K_{c} (3.7) matrix K_+ allows us to describe other Toda lattices associated with several affine root systems [4, 5, 11].

Looking for the dynamical boundary matrix $K(u)$ let us begin with the single operator $L(u)$ (3.1) in the chain and introduce the following ansatz for the dynamical boundary matrix

$$K(u) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(q, u) \tag{3.9}$$

where a, b, c and d are functions of the spectral parameter u and one dynamical variable q only. This matrix $K(u)$ (3.9) depends on half dynamical variables and, therefore, may be constructed from the scalar solutions to the generalized reflection equation.

Inserting the ansatz $K(q, u)$ (3.9) into the dynamical exchange algebras (2.8) and then into the general dynamical equation (2.6) we get two non-trivial upper $c(q, u) = 0$ and lower $b(q, u) = 0$ triangular matrices. In both these solutions the diagonal entries $a(u)$ and $d(u)$ are independent on the dynamical variable q.

Note that the triangular boundary matrices with the property (2.10) are obtained from the dynamical equations (2.6)–(2.8) by using the special ansatz (3.9). Only then, due to the special structure of the boundary matrices, can we see that the transformation (2.9) of the Yang solution $R(u)$ (3.3) is just the Drinfeld twist (2.11) depending on spectral parameters [10] only. Moreover, these twists are closely connected to the twists of the underlying Lie algebra $sl(2)$.

3.1. Lower triangular dynamical matrix

Inserting the lower triangular matrix

$$K_{\mathrm{d}} = \begin{pmatrix} a(u) & 0 \\ c(q, u) & d(u) \end{pmatrix} \tag{3.10}$$

into the system (2.8) one gets two dynamical equations

$$[p, c(q, u)] = -z(q, u)d(u)\exp(q) \qquad [p, c(q, v)] = z'(q, v)a(v)\exp(q) \tag{3.11}$$

related with the following matrices

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ z(q,u) & 0 & 0 & 1 \end{pmatrix} \qquad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ z'(q,v) & 0 & 0 & 1 \end{pmatrix}.$$

Here $z(q,u)$ and $z'(q,v)$ are functions of the spectral parameters and of the dynamical variable q. By using the generators $\boldsymbol{h}, \boldsymbol{e}, \boldsymbol{f}$ of the underlying Lie algebra $sl(2)$

$$[\boldsymbol{h}, \boldsymbol{e}] = 2\boldsymbol{e} \qquad [\boldsymbol{h}, \boldsymbol{f}] = -2\boldsymbol{f} \qquad [\boldsymbol{e}, \boldsymbol{f}] = \boldsymbol{h} \tag{3.12}$$

let us introduce an appropriate element $\mathcal{F} \in U(sl(2)) \otimes U(sl(2))$

$$\mathcal{F}_z = \exp(z \cdot \boldsymbol{f} \otimes \boldsymbol{f}) \qquad z \in \mathbb{C}$$

belonging to a tensor product of the corresponding universal enveloping algebras $U(sl(2))$ [12]. In the fundamental spin-1/2 representation $\rho_{\frac{1}{2}}$ we have

$$F = (\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}})\mathcal{F}_z \qquad G = (\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}})\mathcal{F}_{z'}.$$

However, now z and z' are the proper functions of both spectral parameters, the associated twist (2.9)

$$S(u,v) = FR(u+v)F_{21}^{-1} = I_{zz'}[(u+v)I - \mathrm{i}\eta P]$$
$$I_{zz'} = \exp((z+z') \cdot \boldsymbol{f} \otimes \boldsymbol{f})$$

is equal to

$$S(u,v) = FR(u+v)G = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g & h & 0 \\ 0 & h & g & 0 \\ [z(q,u) + z'(q,v)] & 0 & 0 & f \end{pmatrix}(u,v).$$

Next, the only possible solution to the dynamical equations (3.11) is given by

$$K_\mathrm{d} = \begin{pmatrix} u & 0 \\ u\gamma(q) & -u \end{pmatrix}.$$

Here the entry $\gamma(q)$ is defined by

$$[p, \gamma(q)] = z(q)\exp(q) \tag{3.13}$$

with an arbitrary function $z(q)$ of the dynamical variable q and $z(q) = -z'(q)$, hence $S(u,v) = R(u+v)$.

Next, we have to solve the general dynamical equation (2.6). Let the matrix $S(u,v)$ equal

$$S(u,v) = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g & h & 0 \\ 0 & h & g & 0 \\ z(q,u,v) & 0 & 0 & f \end{pmatrix}(u,v)$$

where $z(q;u,v)$ is an arbitrary entry. Substituting this matrix and the matrix K_d (3.10) into (2.6) one gets the following dynamical equation

$$a(v)[p, c(q,u)] + d(u)[p, c(q,u)] = z(q;u,v)\frac{a(v)d(u)}{u+v-\mathrm{i}\eta}\exp(q) \tag{3.14}$$

which by $c(q,u) = \gamma(q)u$ reads as

$$[p, \gamma(q)] = z(q;u,v)\frac{a(v)d(u)}{(ua(v) + vd(u))(u+v-\mathrm{i}\eta)}\exp(q). \tag{3.15}$$

Solving the generalized reflection equation (2.7) with this matrix $S(u, v)$ one gets the same possible non-trivial solution K_{d} (3.10) only.

To construct the transfer matrix $\tau(u)$ (2.14) let us substitute $q = q_1$ in the matrix $K_{\mathrm{d}}(q, u)$ and $q = -q_n$ in the 'conjugated' matrix $K_+(q, u) = K_{\mathrm{d}}^{\mathrm{t}}(-q_n, u)$. Thus, the dynamical boundary matrices K_{d} and $K_{\mathrm{d}+}$ have common quantum spaces with the first L_1 and with the last L_n operators in the lattice. This generating polynomial

$$\tau(u) = \mathrm{tr}\left(K_{\mathrm{d}}^{\mathrm{t}}\left(-q_n, u + \frac{\mathrm{i}\eta}{2}\right)L_n(q_n, u)\ldots L_1(q_1, u)\right.$$
$$\left.\times K_{\mathrm{d}}\left(q_1, u - \frac{\mathrm{i}\eta}{2}\right)\sigma_2 L_1^{\mathrm{t}}(q_1, -u)\ldots L_n^{\mathrm{t}}(q_n, -u)\sigma_2\right)$$

has the form

$$\tau(u) = H_1 u^{2n} + H_2 u^{2n-2} + \cdots$$

and gives rise to the commutative family of n functionally independent integrals of motion. If the entry $z(u, v, q)$ is independent on the dynamical variable q then the solution to equations (3.13) and (3.15) is equal to

$$\gamma(q) = \gamma\exp(q) + \beta$$

whereas the first integral H_1 has the following factorable form

$$H_1 = J_1 \cdot J_n = (2p_1 + \gamma_-\,\mathrm{e}^{2q_1} + \beta_-\,\mathrm{e}^{q1})\,\mathrm{e}^{q_1}\cdot\,\mathrm{e}^{-q_n}(-2p_n + \gamma_+\,\mathrm{e}^{-2q_n} + \beta_+\,\mathrm{e}^{-qn}).$$

Here (γ_-, β_-) and (γ_+, β_+) are the free parameters associated with the boundary matrices $K_{\mathrm{d}}(q_1, u)$ and $K_{\mathrm{d}}^{\mathrm{t}}(-q_n, u)$, respectively.

This integrable system may be considered as the constrained Hamiltonian system either with one constraint $H_1 =$ constant, or with two constraints

$$q_1 = \text{constant}_1,\ \ q_n = \text{constant}_n \qquad \text{or} \qquad J_1 = \text{constant}_1,\ \ J_n = \text{constant}_n.$$

3.2. Upper triangular dynamical matrix

Inserting the upper triangular matrix

$$K_{\mathrm{D}} = \begin{pmatrix} a(u) & b(q, u) \\ 0 & d(u) \end{pmatrix} \tag{3.16}$$

into the system (2.8) one gets two dynamical equations

$$[p, b(q, u)] = -w(q, u)\,a(u)\exp(q) \qquad [p, b(q, v)] = -w'(q, v)\,d(v)\exp(q) \tag{3.17}$$

related to the following matrices

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & w(q, u) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & w'(q, v) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.18}$$

Here $w(q, u, v)$ and $w'(q, u, v)$ are functions of spectral parameters and the dynamical variable q.

In generators (3.12) the corresponding twist element $\mathcal{F} \in U(sl(2)) \otimes U(sl(2))$ is equal to

$$\mathcal{F}_w = \exp(w \cdot \boldsymbol{f} \otimes \boldsymbol{e}) \qquad w \in \mathbb{C}$$

(see the factorization of the universal R-matrix in [12, 16]). In the fundamental spin-1/2 representation $\rho_{\frac{1}{2}}$ we have

$$F = (\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}})\mathcal{F}_w \qquad G = P(\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}})\mathcal{F}_{w'}P.$$

However, now w and w' are the corresponding (3.17) functions of the spectral parameters. We can see, $G = F_{21}^{-1}$ up to change the twist parameters $w(q,u) \to -w'(q,v)$. The associated twisted matrix $S(u,v)$ (2.9) is equal to

$$S(u,v) = F_{12}(w)((u+v)I - \mathrm{i}\eta P)F_{21}^{-1}(w') = J_{ww'} - \mathrm{i}\eta I_{ww'}P$$

where

$$J_{ww'} = (\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}})\exp(w \cdot \boldsymbol{f} \otimes \boldsymbol{e} + w' \cdot \boldsymbol{e} \otimes \boldsymbol{f})$$
$$I_{ww'} = (\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}})\exp((w + w') \cdot \boldsymbol{f} \otimes \boldsymbol{e}).$$

In fact, matrix $S(u,v)$ has the following form

$$S(u,v) = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g + h(w+w') + ww'g & h + wg & 0 \\ 0 & h + w'g & g & 0 \\ 0 & 0 & 0 & f \end{pmatrix}(u,v) \tag{3.19}$$

where f, g and h are entries of the initial R-matrix (3.3).

For a given S-matrix (3.19) we shall not solve the dynamical equations (3.17) and the generalized reflection equation (2.7) generically. Here, we restrict ourselves to those particular solutions which are related to interesting physical systems only.

At first, let us introduce an upper triangular matrix K_{D} with the following entries

$$a(u) = d(u) = u \qquad b(q,u) = \beta\exp(q) + \gamma. \tag{3.20}$$

One immediately gets

$$w(u) = \mathrm{i}\eta\beta u^{-1} \qquad w'(v) = \mathrm{i}\eta\beta v^{-1}$$

where the parameters w and w' of the twist are independent on the dynamical variable q. Thus, from the linear matrix-function $R(u)$ (3.3) we obtain the rational matrix $S(u,v)$, which has the upper and lower triangular residues at the points $u = 0$ and $v = 0$, respectively.

The second more complicated solution $K_{g\mathrm{D}}$ is defined by

$$a(u) = u^2 + \alpha u + \delta \qquad d(u) = a(-u) \qquad b(q,u) = (\beta\exp(q) + \gamma)u. \tag{3.21}$$

For both solutions

$$K(u)K(-u) = \mp\phi(u)I$$

where the function $\phi(u)$ is equal to the determinant of the matrices K_{D} and $K_{g\mathrm{D}}$, respectively.

To construct the transfer matrix $\tau(u)$ (2.14) let us substitute $q = q_1$ in the matrix $K_{\mathrm{D}}(q,u)$ and $q = -q_n$ in the 'conjugated' matrix $K_+(q,u) = K_{\mathrm{D}}^{\mathrm{t}}(-q_n,u)$. This means that the dynamical boundary matrices K_{D} and $K_{\mathrm{D}+}$ have common quantum spaces with the first L_1 and the last L_n operators in the lattice. The generating polynomial $\tau(u)$ (2.14)

$$\tau(u) = \mathrm{tr}\bigg(K_{\mathrm{D}}^{\mathrm{t}}\bigg(-q_n, u + \frac{\mathrm{i}\eta}{2}\bigg)L_n(q_n,u)\ldots L_1(q_1,u)$$
$$\times K_{\mathrm{D}}\bigg(q_1, u - \frac{\mathrm{i}\eta}{2}\bigg)\sigma_2 L_1^{\mathrm{t}}(q_1,-u)\ldots L_n^{\mathrm{t}}(q_n,-u)\sigma_2\bigg)$$

has the form

$$\tau(u) = H_1 u^{2n+2} + H_2 u^{2n} + \cdots \tag{3.22}$$

and gives rise to the commutative family of $(n+1)$ functionally independent integrals of motion. The use of the second dynamical matrix $K_{g\mathrm{D}}$ (3.21) leads to similar results.

For both solutions the first integral H_1 in expansion (3.22) may be considered as the factorable constraint

$$H_1 = J_1 \cdot J_n = (-2\,\mathrm{e}^{q_1} + \beta_-\,\mathrm{e}^{q_1} + \gamma_-) \cdot (2\,\mathrm{e}^{-q_n} + \beta_+\,\mathrm{e}^{-qn} + \gamma_+).$$

Here (γ_-, β_-) and (γ_+, β_+) are the free parameters associated with the dynamical entries of the boundary matrices $K_\mathrm{D}(q_1, u)$ and $K_{\mathrm{D}+} = K_\mathrm{D}^\mathrm{t}(-q_n, u)$, respectively.

In contrast with the lower triangular solution this constraint is easy removed, if the parameters w and w' of the twist are independent on the dynamical variable q. Namely, unless otherwise indicated, set

$$\beta_\pm = \mp 2$$

such that $H_1 = \gamma_-\gamma_+ =$ constant and the generating polynomial (3.22) gives rise to n independent integrals of motion only.

After canonical transformation of the variables

$$\mathrm{e}^q \to 1 - ch(q) \tag{3.23}$$

in the first, $\mathcal{H}_1$ and in the last, $\mathcal{H}_n$ local quantum spaces in the chain, the associated Hamiltonians H read as

$$\begin{aligned} H_\mathrm{D} &= H_\mathrm{A} + \exp(-q_{n-1} - q_n) + \exp(q_1 + q_2) \\ H_{gD} &= H_\mathrm{D} + \frac{\alpha_1}{\sinh^2(q_1/2)} + \frac{\alpha_2}{\sinh^2 q_1} + \frac{\alpha_3}{\sinh^2(q_n/2)} + \frac{\alpha_4}{\sinh^2 q_n}. \end{aligned} \tag{3.24}$$

Here the four constants α_j are functions of the four initial constants $\alpha_\pm$ and $\delta_\pm$ in the diagonal entries (3.21) of the boundary matrices [5].

Thus, the dynamical boundary matrix K_D (3.20) corresponds to the Toda lattices associated with the root system $\mathcal{D}_n$ [11]. Hence, the single spin-1/2 representation $T(u)$ (2.12) and (3.1) of the Yangian $Y(sl(2))$ may be used to construct the monodromy matrices for the Toda lattices associated to all the classical infinite series of root systems. The second solution $K_{g\mathrm{D}}$ (3.21) allows us to add four extra parameters in the Hamiltonian H_D and this is known as Inozemtsev's generalization of the Toda system [13].

The boundary matrices $K'_\mathrm{D} = L(u)K_\mathrm{D}L^{-1}(-u)$ and $K'_{g\mathrm{D}} = L(u)K_{g\mathrm{D}}L^{-1}(-u)$ were at first found in [5] starting from the known $(2n \times 2n)$ Lax matrices [13]. They are solutions to the usual reflection equation, which have the ultralocal commutation relations with other matrices in the chain. Note, we have to use the two different representations $T(u)$ (2.12) of the Yangian $Y(sl(2))$ to describe Toda lattices associated with the $\mathcal{BC}_n$ and $\mathcal{D}_n$ root systems by using the matrices K_c and K'_D, respectively. In classical mechanics factorization $K'_\mathrm{D} = L(u)K_\mathrm{D}(q, u)L^{-1}(-u)$ on the terms with non-ultralocal commutator relations has been applied to the separation of variables in [15].

Two outer automorphisms of the space of infinite-dimensional representations of the Lie algebra $sl(2)$ are used to recover these boundary matrices K'_D and $K'_{g\mathrm{D}}$ in [14]. It would be interesting to study the interrelations among these automorphisms of $sl(2)$ and the twists (2.11) of the usual rational R-matrix.

The relativistic Toda lattices associated with the $\mathcal{D}_n$ root systems [19] may be easily embedded in the proposed scheme as well. In this case the corresponding R-matrix is the known trigonometric solution to the Yang–Baxter equation [19] and the associated twist is connected to the twist of the algebra $sl_q(2)$.

3.3. The Drinfeld twist and separation of variables method

We know that the representation theory of Drinfeld twists may be very useful in the framework of the algebraic Bethe ansatz [16]. According to [16], for the XXX-$\frac{1}{2}$ and XXZ-$\frac{1}{2}$ Heisenberg (inhomogeneous) quantum spin chains of finite length n the associated $\mathcal{F}$-matrices diagonalize the generating matrix of scalar products of the quantum states of these models. They also diagonalize the diagonal (operator) entries of the quantum monodromy matrix.

Now, we discuss the interrelations of the Drinfeld twists and the separation of variables method proposed by Sklyanin [17]. For the sake of brevity we shall work with the corresponding classical objects.

Starting from the known $(2n \times 2n)$ Lax matrices we can obtain solutions K'_{D} or $K'_{g\mathrm{D}}$ to the reflection equation (2.2) with ultralocal commutation relations. In classical mechanics the associated (2×2) Lax matrix is equal to

$$\begin{aligned}\mathcal{L}'(u) = K'^{\mathrm{t}}_{\mathrm{D}}(-q_n, u) L_{n-1}(q_{n-1}, u) \ldots L_2(q_2, u) \\ \times K'_{\mathrm{D}}(q_1, u)\sigma_2 L_2^{\mathrm{t}}(q_2, -u) \ldots L_{n-1}^{\mathrm{t}}(q_{n-1}, -u)\sigma_2.\end{aligned} \tag{3.25}$$

Here all the local matrices $L_j(u)$, $j = 2, \ldots, n-1$ and boundary matrices $K'_{\mathrm{D}}(q_{1,n}, u)$ are defined on the different phase spaces such that $\{\overset{1}{L}_j(u), \overset{2}{K'_D}(q_{1,n}, v)\} = 0$.

In the Sklyanin approach [17] the use of this Lax matrix forces application of the dynamical normalization of the corresponding Baker–Akhiezer vector-function Ψ [15]. The choice of the proper normalization α [17]

$$(\boldsymbol{a}, \Psi) = \sum_{j=1} \alpha_j(u)\Psi_j(u) = 1$$

allows us to fix special analytical properties of this meromorphic eigenfunction Ψ of the Lax matrix $\mathcal{L}'(u)$

$$\mathcal{L}'(u)\Psi = z\Psi.$$

An appropriate normed vector-function Ψ has to possess the necessary number of poles in involution and all the extra poles of Ψ are constants [17]. However, one does not usually know the separating normalization in advance.

It is clear, using a similarity transformation for the Lax matrix $\mathcal{L}'(u)$

$$\mathcal{L}'(u) \to \mathcal{L}(u) = V(u)\mathcal{L}'(u)V^{-1}(u) \tag{3.26}$$

that any normalization $\boldsymbol{a}$ may be turned into the simplest constant normalization vector [17]. So, in classical mechanics the problem is to find an appropriate similarity transformation (3.26).

In quantum mechanics an action of the similarity transformations (3.26) may be transferred into the R-matrix level. If we transform the R-matrix in the special twists only, we essentially restrict the freedom related to similarity transformations (3.26). The twisted R-matrix possesses most of the properties of the usual R-matrix [10] and possibly gains some new properties, for example admissible [10] or factorizing [16] twists may be used.

In this paper, starting from the known Yang solution $R(u)$ of the Yang–Baxter equation we introduce the twist $S(u, v)$, which has other analytical properties. By next solving the corresponding dynamical equations and the generalized reflection equation we obtain the dynamical boundary matrices K_{D} or $K_{g\mathrm{D}}$. The associated (2×2) Lax matrix is equal to

$$\mathcal{L}(u) = K^{\mathrm{t}}_{\mathrm{D}}(-q_n, u) L_n(q_n, u) \ldots L_1(q_1, u) \times K_{\mathrm{D}}(q_1, u)\sigma_2 L_1^{\mathrm{t}}(q_1, -u) \ldots L_n^{\mathrm{t}}(q_n, -u)\sigma_2. \tag{3.27}$$

Here two local matrices $L_{1,n}(u)$ and boundary matrices $K_{\mathrm{D}}(q_{1,n}, u)$ are defined on the common phase spaces and the corresponding Poisson brackets relations $\{\overset{1}{L}_{1,n}(u), \overset{2}{K}_D(q_{1,n}v)\} \neq 0$ may be derived from the quantum algebras (2.8).

Two Lax matrices $\mathcal{L}'(u)$ (3.25) and $\mathcal{L}(u)$ (3.27) are related by the similarity transformation (3.26). Now $V(u)$ is equal to the inversion matrix $V(u) = L_n^{\mathrm{i}}(q_n, -u)$ (3.5). This transformation leads to the appearance of the twist $S(u, v)$ (3.19) instead of the Yang solution $R(u + v)$ in the quantum algebraic relations.

Although the matrices K_{D} and $K_{g\mathrm{D}}$ are solutions to the more complicated generalized reflection equation (2.7) and the dynamical equations (2.6), nevertheless these matrices make it possible to use the simplest constant normalization of the associated Baker–Akhiezer function [17] in classical mechanics. Namely, for the Toda lattice associated to the $\mathcal{D}_n$ root system in classical mechanics the Lax matrix $\mathcal{L}(u)$ (3.27) has the following matrix form

$$\mathcal{L}(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}(u). \tag{3.28}$$

Choosing the simplest constant normalization $\boldsymbol{\alpha} = (1, 0)$ of the associated Baker–Akhiezer function [17] the separation variables $\{x_j\}_{j=1}^n$ are defined as zeros of the entry $B(u)$ [15]

$$B(u = x_j) = 0$$

according to the general recipe [17]. It is easy to prove that they are real eigenvalues of the symmetric matrix defined by recursion, which has been proposed in [5]. Other separation variables are sitting on the spectral curve of the Lax matrix (3.28) with the previously defined variables $\{x_j\}_{j=1}^n$ [15, 17].

Thus, in the considered example, the use of Drinfeld twists in the quantum case leads to a suitable Lax representation and to simple separating normalization in classical mechanics. We see that the algebraic properties of the Lax matrix relate to the analytical properties of its eigenfunction. Of course, we have to study suitable properties of this twist, which correspond to the separation of variables. Note, the constant normalization of the Baker–Akhiezer function allows us to develop the quantum counterpart of the classical separation of variables method within the R-matrix approach [18] for Toda lattices.

4. Integrable tops closed to the Toda lattice

Let the variables l_i, g_i, $i = 1, 2, 3$ be generators of the Lie algebra $e(3)$ with commutator relations

$$[l_i, l_j] = -\mathrm{i}\eta e_{ijk} l_k \qquad [l_i, g_j] = -\mathrm{i}\eta e_{ijk} g_k$$
$$[g_i, g_j] = 0 \qquad i, j = 1, 2, 3$$

and with the following Casimir operators

$$J_1 = (g, g) \qquad J_2 = (l, g).$$

Let us introduce the quantum operator $T(u)$ for the Neumann system

$$T(u) = \begin{pmatrix} u^2 - 2l_3u - l_1^2 - l_2^2 - \frac{1}{4} & \mathrm{i}(g_+u - \frac{1}{2}\{g_3, l_+\}) \\ \mathrm{i}(g_-u - \frac{1}{2}\{g_3, l_-\}) & g_3^2 \end{pmatrix} \tag{4.1}$$

where braces $\{,\}$ mean an anticommutator. The operator $T(u)$ (4.1) at the level $J_2 = (l, g) = 0$ obeys the fundamental commutator relations (2.1) with the rational R-matrix (3.3) and closely related to the Toda system [20].

The use of the usual covariance property (2.3), operator $T(u)$ (4.1) and the constant boundary matrices K_c (3.7) allows us to describe the quantum Kowalewski–Chaplygin–Goryachev top [20].

Let us consider the known constant solution (3.20) or (3.21) to the generalized reflection equation (2.7) with the twisted matrix $S(u,v)$ (3.19). By using this solution we may introduce another solution of the same equation (2.7), which is the function defined on the Abel subalgebra of $e(3)$. Thus, we obtain dynamical boundary matrices on the common quantum space with operator $T(u)$ (4.1)

$$K_- = \begin{pmatrix} u^2+\alpha_- u+\delta_- & \mathrm{i}u(\beta_- g_+ + \gamma_-) \\ 0 & u^2-\alpha_- u+\delta_- \end{pmatrix}$$
$$K_+ = \begin{pmatrix} u^2+\alpha_+ u+\delta_+ & 0 \\ \mathrm{i}u(\beta_+ g_- + \gamma_+) & u^2-\alpha_+ u+\delta_+ \end{pmatrix} \tag{4.2}$$

such that

$$\overset{1}{K}_+ \overset{2}{K}_- = \overset{2}{K}_- \overset{1}{K}_+ .$$

If we set

$$\beta_\pm = -2$$

then the generating polynomial (2.14) is equal to

$$\tau(u) = \mathrm{tr}[K_+ T(u) K_- \sigma_2 T^{\mathrm{t}}(-u)\sigma_2] = u^6(2J_1 - \gamma_+\gamma_-) + u^4 J_3 + u^2 J_4.$$

Two independent integrals of motion J_3 and J_4 mutually commute at the level $J_2 = (l, g) = 0$. Here we present the corresponding Hamiltonian J_3 in classical mechanics only

$$\begin{aligned} J_3 = {} & l_+l_-(g_+\gamma_+ - \gamma_+\gamma_- + g_-\gamma_-) - 2l_3^2(2g_- - \gamma_+)(2g_+ - \gamma_-) \\ & + 2l_3[2J_1(\alpha_+ + \alpha_-) - (\alpha_-\gamma_+ g_+ + \alpha_+\gamma_- g_-)] \\ & + l_3 g_3[2l_-(2g_+ - \gamma_-) + 2l_+(2g_- - \gamma_+) - J_3(\alpha_+ + \alpha_-)] \\ & + g_3(\gamma_+\alpha_- l_+ + \alpha_+\gamma_- l_-) + \delta_+\gamma_- g_- + \delta_-\gamma_+ g_+. \end{aligned}$$

By analogy with (3.23), the use of automorphisms of the Lie algebra $e(3)$ might allow us to rewrite this Hamiltonian in a more physical form. However, for us it is more important that all three matrices in the chain are defined in single common quantum space.

5. Conclusions

We have discussed the dynamical boundary matrices, which act in common quantum space with other operators in the chain. These matrices are solutions to the generalized reflection equation and dynamical exchange equations. This system of equations includes the usual R-matrix and its Drinfeld twist depending on spectral parameters. To construct the Drinfeld twists we use known twists of the underlying Lie algebras. In the framework of the proposed method non-trivial representations of the reflection equation algebra may be produced in a systematic way. As an example, we consider the twist of the Lie algebra $sl(2)$ related to the Toda lattices associated to the $\mathcal{D}_n$ root system.

We believe that it is possible to apply simple algebraic tools from the theory of Lie algebra to construct new representations of R-matrix algebra [14]. Moreover, the different properties of the associated integrable models may be implicitly related to their underlying algebraic properties. As an example, it is very interesting to compare the relationship between the Drinfeld twist theory and the separation of variables method proposed by Sklyanin [17].

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